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SOME SUGGESTIONS FOR TEACHING GEOMETRY TO DEVELOP CLEAR THINKING

Gilbert Ulmer

This Issue of Kansas Studies in Education

There has been an increasing amount of interest among high school mathematics teachers the past few years in teaching geometry to develop clear thinking. It used to be thought that students would become more logical in all of their thinking as a direct result of their contact with the logic of geometry. Although this belief is no longer generally accepted, there are many teachers who believe that the study of geometry does offer unusual opportunities for studying several principles of clear thinking. They believe that direct attention should be given to those principles and to their application in various types of situations.

Much has been written about the desirability of using non-geometric material in the classroom to focus pupils' attention upon principles of clear thinking and to show the operation of those principles in both geometric and non-geometric situations. Many teachers who would like to use some non-geometric material in this manner have found that there is little such material available. The present issue of *Kansas Studies in Education* is distributed to high school mathematics teachers with the hope that they may find it of some use as a source of classroom materials and suggestions for teaching geometry to develop an understanding of principles of clear thinking.

Suggestions For An Introductory Unit

Explain to the pupils at the very beginning that *the most important purpose of the course is to get them to be more critical of their thinking.* Tell them that in this course they will also learn a great many things about geometric figures like circles and triangles and designs and patterns of many kinds, but that it will be even more important that they *learn how to think straight.* Point out the need for clear and accurate thinking by means of a discussion similar to the following:

> "One of the most important needs of the world today is the need for clear thinking—and there is not much of it being done. People just drift along without thinking things through very carefully. For the most part they are content to let a few people do their thinking for them and quite often those few are not thinking of the common good but of their own selfish interests. A noted writer said recently that most adults have the mentality of children. A lawyer, in a recent article in Scribners, said that people can't analyze evidence, that they do not understand what is proof and what is not. This is because most people have never learned how to think. Now in this course we are going to study the processes of clear and logical thinking. We want you to reach the place where you won't be so likely to be blinded by misleading advertising and where you will see through the flimsy by high-sounding arguments so often used by politicians and others who are trying to influence us.

"We are going to start at the very beginning in this matter of logical thinking, and the first thing we have to do is to learn how certain statements *necessarily imply* others. Here's a very easy example to show what this means. Let's consider these two statements. (Write on board.)

(1) All Chinese are yellow.
(2) Wing Toy is Chinese.

Do you see another statement that must follow as a necessary result of these two?"

The pupils will volunteer the third statement readily. Make sure the class understands that if we accept these first two statements, then we *must* accept the third statement. They ought to see that we haven't any choice in the matter—the third statement is implied in the first two without our having anything to do with it. Another way to put this is to say that the third statement really is *contained* in the first two.

It may be interesting at this point to change the second statement in the above example to read, "(2) Joy Lee is yellow," and to ask the class again if there is another statement which must follow as a necessary result of these two. Some pupil probably will volunteer the statement, "Joy Lee is Chinese." If not, the teacher should propose it. The class should see that this third statement is *not* contained in the first two. It may or may not be true; the point is, it does not follow as a necessary consequence of the first two.

Have the class consider another example such as the following:

(1) All members of the drum corps are at least six feet tall.
(2) Henry Williams is a member of the drum corps.

Ask the pupils to tell what third statement we are forced to accept if we agree to accept these two.

If time permits, several other examples may be considered in class and still others may be dictated to be studied for the next day. In some of the examples two statements should be given and the students required to determine whether or not some third statement must follow as a result of the two given statements; in other examples three statements should be given and the students required to decide whether or not the third statement in each case follows as a necessary consequence of the first two.

The following list may be suggestive. It is recommended that modification be made wherever possible in the examples to make them fit better your own school or community. It is also suggested that you make up additional exercises using topics of current interest. Be sure that the students understand that they are not to be concerned at all with the validity of the first two statements—in fact they may not agree with them at all; their problem just now is to determine whether or not these two statements lead necessarily to a third statement.

All new students in our school are required to take an examination.
Bob is a new student in our school.

All 90° angles are right angles.
This angle is a 90° angle.

No one but boys who are six feet tall can belong to the 6-footer club.
John is six feet tall.

All seamen like to fight.
Murphy is a seaman.

All freshmen must enroll before going to class.
Mary enrolled before she went to class.

A horse is a four-legged animal.
This is a horse.

All of the members of the orchestra went on a picnic today.
Alfred is a member of the orchestra.

Clarke's thread will not break.
This thread will not break.

All fruit trees bear fruit.
This tree does not bear fruit.

Only blackbirds have red wings.
This bird has red wings.

Everyone in the English class had his lesson today.
John had his lesson today.

All cars are dangerous.
A Buick is a car.
A Buick is dangerous.

All brown-haired girls are pretty.
Dorothy is a brown-haired girl.
Dorothy is pretty.

All snowflakes are hexagonal in shape.
This flake or crystal is hexagonal in shape.
This is a snowflake.

Homemade ice cream is richer than ice cream bought at the store.
This is homemade ice cream.
This ice cream is richer than ice cream bought at the store.

The rooster crows at exactly six o'clock every morning and at no other time.
The rooster is crowing now.
It must be six o'clock in the morning.

All ice men are tall.
William is a tall man.
William is an ice man.

September has thirty days.
This month has thirty days.
This month is September.

A boy must weigh at least 140 pounds before he can play football.
Bob plays football.
Bob weighs at least 140 pounds.

No one but first graders had a holiday today.
Betty had a holiday today.
Betty is a first grader.

Every Monday is wash day at our house.
This is wash day at our house.
This is Monday.

Silhouettes are black and white pictures.
This picture is black and white.
This picture is a silhouette.

Only the members of the senior class had their pictures taken today.
Mary had her picture taken today.
Mary is a member of the senior class.

All school houses are red.
This building is a school house.
This building is red.

Only hot days are disagreeable.
Wednesday was a disagreeable day.
Wednesday was a hot day.

All good swimmers may enter the swimming meet.
Jack is a good swimmer.
Jack may enter the swimming meet.

All taxi drivers are careful drivers.
Dick is a taxi driver.
Dick is a careful driver.

Snow is on the ground only in the winter season.
Snow is on the ground now.
This is the winter season.

All Japanese have slanting eyes.
Ching has slanting eyes.
Ching is a Japanese.

Every book in the library is well bound.
"Little Women" is a book in the library.
"Little Women" is a well bound book.

All girls with dark hair have brown eyes.
Betty has dark hair.
Betty has brown eyes.

Only members of the sophomore class knew of the picnic.
Mary knew about the picnic.
Mary is a member of the sophomore class.

All Skrip is good ink.
This is good ink.
This ink is Skrip.

All of the men who work in this building begin work at 8 o'clock.
Albert works in this building.
Albert begins work at 8 o'clock.

All court houses are square brick buildings with clocks.
This is a square brick building with a clock.
This building is a court house.

It has been found very profitable for the students to make up examples of their own to bring to class for discussion. This might be a part of the assignment for the second day if you feel on the first day that the pupils are ready to do this. Have them list their three statements and write in "yes" or "no" beside the third one. *Avoid the use of the words* "true" *and* "false", *since we are not conecrned with the truth of the statements. We want the pupil to understand that he is merely to answer the question, "Does this third statement have to follow as a necessary result of the first two?"*

On the second day you might review the need for careful and accurate thinking somewhat along the following lines:

> "There is too much loose thinking in the world today. Too often, people reach decisions on the basis of prejudice rather than as a result of clear thinking. People are impulsive. We see people doing silly things and getting themselves into tight places all the time just because they do not think.
>
> "Yesterday I told you that in this course we are going to study the ways in which careful and accurate thinking may be done. These exercises which we have for today may seem extremely simple, but I told you that we were going to start at the very beginning. It is absolutely necessary that you be able to tell whether or not certain statements imply other statements."

Discuss with the class the exercises that have been assigned. A number that have been made up by the pupils might be put on the board for discussion. Also have the pupils practice changing an exercise like the following, in which the third statement does follow, to obtain one in which a third statement does not necessarily follow.

(1) All red-haired people have bad tempers.
(2) William has red hair.
(3) William has a bad temper.

This might be changed into the following:

(1) All red-haired people have bad tempers.
(2) William has a bad temper.
(3) William has red hair.

The pupils might be asked to go through a list of these examples and make slight changes in the first two statements in such a way as to alter the conclusions.

Next, have the class consider examples in which only a part of the argument is given and have them build the rest of it. For instance, call their attention to the following statement, which is a common kind: "This is a Spaulding tennis racket so it must be a good one." Ask what assumption would have to be made—what statement would have to be taken for granted—in order to insure the conclusion contained in the above statement. After this question has been answered, the class should put the argument into the following form:

(1) All Spaulding tennis rackets are good rackets.
(2) This is a Spaulding tennis racket.
(3) This must be a good racket.

Another example is: "This must be good music; it's grand opera." The class should first decide what the conclusion is, and then find an assumption which would make this conclusion necessary. The argument should finally be stated as follows:

(1) All grand opera is good music.
(2) This is grand opera.
(3) This must be good music.

Tell the pupils that they will find many examples of this kind in advertising matter and ask them to be on the lookout for them and to make lists of them to bring to class. An example taken from advertising matter is: "Only Pepsodent Tooth Powder contains irium." There is no conclusion stated but undoubtedly the advertiser wanted us to conclude—that is he wanted us to have the feeling, somehow—that Pepsodent is the best tooth powder. After agreeing on the conclusion that we were meant to draw, the class should find what assumption it would be necessary to make in order that this conclusion must follow.

The list of examples given below may be of use in getting the pupils started to finding examples by themselves. In considering such examples, have the pupils determine the conclusion first and then construct the rest of the argument so that it leads to that conclusion. Some common sources of material of this kind are the following:

Everyday conversation
—around school
—on the street
—at home

Advertising matter
—in newspapers and magazines
—over the radio
—on signs in stores
—on billboards

Newspaper and magazine articles

Speeches

List of examples:

"Mary must be smart. She gets A's in her school work."
"It must not have hailed here, because the wheat is not beaten down."
"Mr. Smith must be a religious man, because he goes to Church every Sunday."
"Bayer's aspirin is the original aspirin."
"None of the guests said a word about the pretty flowers. I guess they didn't notice them."
"The Dionne quintuplets have never used any soap but Palmolive."
"The fudge didn't get hard, I must not have cooked it long enough."
"Sanfords blue-black ink is pen tested."
"There was no light in his window so I knew he had gone."
"He is a musician, so he probably has long hair."

"That must be a good baseball team; they have good pitchers."

"The tubes and the connections are O.K., so the trouble must be in the condenser."

"Folgers is freshly roasted coffee."

"No, her hair isn't naturally curly. Haven't you noticed how it gets straight when it is damp?"

"This substance is lighter than wood; therefore it will surely float in water."

"Mr. Jones voted for Roosevelt, so he must be a democrat."

"A larger per cent of children of elementary school age attend school in the United States than is the case in any other country of the world. Therefore the United States has the best elementary school system in the world."

A Kreml ad shows a picture of a young man who is bald and prints the warning, "Don't let this happen to you."

Along with examples of this kind consider the conclusions which advertisers want us to draw from pictures. A beautiful girl is shown smoking a certain brand of cigarette. The advertiser hopes we will conclude that all beautiful girls or all smartly dressed people smoke this brand of cigarette. The same kind of appeal is made in ads for clothing, soaps, cars, etc. Try to develop in the pupils an awareness of the ways in which advertisements are made to appeal to people.

During the next few days have the pupils bring in and consider many examples similar to those in the above list. (In fact, it would be well to encourage them to keep this up throughout the year.) At the same time begin some work in informal geometry. It is recommended from the standpoint of interest and also as a means of preparing the pupils for definitions and assumptions which we shall want them to make soon, that this informal work be begun by having the pupils become acquainted with the use of their rulers and compasses without formal or detailed directions. Have them make or copy any designs which appeal to them and which they think they can draw. Don't tell them how to make geometric constructions but get them to use their imagination and make some designs or patterns. Usually pupils like this and it puts them in a better position to study relationships between lines, angles, triangles, arcs of circles, and other simple geometric figures, because they can see these things as parts of whole figures rather than as so many isolated entities. Next, proceed from the whole to its parts by explaining that in order to understand very much about patterns and designs like those the pupils have just made it will be necessary to see how the different parts are made and put together. Then have the class consider the fundamental constructions, without proofs and with definitions brough in rather informally only as they are needed to describe or explain concepts with which the pupil has already become familiar.

Near the end of this work on informal constructions have the pupils consider an exercise like the following*:

"Suppose we assume the following statements to be true:

(1) All employees of the White Steel Company are members of the Allied Workers Union.

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* This exercise is patterned after one used by Professor U. G. Mitchell in one of his mathematics classes at the University of Kansas.

(2) No company can employ a member of the Allied Workers Union at wages of less than five dollars per day.
(3) No foreigner can be a member of the Allied Workers Union.
(4) William Smith is an employee of the White Steel Company.
(5) William Smith and his brother, John Smith, together earn nine dollars per day wages.

Make as complete a list as you can of other statements that follow as a result of these five statements. Also, indicate by number which statements justify each conclusion. For example, if statements (1) and (2) imply another statement, tell what it is and write (1) and (2) after it. The conclusions also may be numbered and used to obtain still other conclusions."

Point out that this is like the other exercises they have been doing except that they keep repeating the process. This might be given as an assignment for outside work and discussed in class the next day. Statement (5) may cause some differences of opinion. This will be a good opportunity to demonstrate the need for defining our terms, the need for the class to agree on how certain words or expressions are to be interpreted.

Assumptions in Geometry

Explain to the class that this system which they have just built is like a miniature science. Point out that any science consists of a list of assumptions (statements taken for granted) and all of the statements which are the necessary consequences of those assumptions. Tell the pupils that the class is going to build a science of geometry in very much the same way that the little system concerning the manufacturing plant was built. A list of statements will be taken for granted concerning the simpler geometric elements and most of the course will be devoted to finding other statements that must follow as a result of these assumptions.

When assumptions are being made, it frequently happens that members of the class have a feeling that some of the assumptions are trivial and that they are scarcely worth discussing. One way to deal with this is to pick out one or two such assumptions and show their usefulness. For example, in connection with the assumption that the shortest path between two points is a straight line, ask the members of the class if they could form a triangle with any three lengths you might give them for sides. Many of them probably will think they can. Push it a little farther and ask if they mean that they can lay down any three sticks you would give them to form a triangle—and warn them that they must use the full length of each stick. There probably will be some who will still agree it can be done. Then ask them to sketch a triangle with three sides such as 5, 3, and 10. When they see the point, show them by means of a diagram that the assumption, "The shortest path between two points is a straight line," would make it necessary that no side of a triangle be as long as the sum of the other two sides. When they see that the use of this apparently insignificant assumption could have prevented their making the mistake they have just made, they should come to have more respect for some of the statements that at first appear trivial.

As a result of their experience with the exercises in logic the pupils should

be in a position to understand the important role which assumptions are to play in the science of geometry. The assumptions should be discussed in class and the pupils should agree that they are willing to take these statements for granted. They should bear in mind that most of the course will be devoted to examining the consequences of these assumptoins.

A common method of dealing with the first assumptions is that of having the pupils read the list given in the text and learn what they mean. However, the wording of most of the assumptions in the text will seem unnecessarily stiff and formal to the pupils unless they have discussed them first. We suggest that at least some of the assumptions be reached as a result of class discussion. Have the pupils word them. Their first attempts probably will be clumsy and not absolutely accurate. Lead them to see this and help them to refine their statements. Then, finally, get them to regard the list of statements in the text as a record of the work that they have done. *In fact, a good general principle for the teacher to follow throughout the entire course would be that of making the textbook follow the class rather than having it lead the class.* Let the statements in the text sum up and state in a concise and accurate form the principles or relationships which have been considered by the class. It is only after the pupil has had some experience with these concepts that he is ready for the refinements found in the statements in the text. If he is not provided with opportunities to work his own way through to these refinements the pupil probably will memorize statements without understanding much of their true significance.

At some point in this early work on assumptions the Declaration of Independence ought to be considered—especially the first part of the second paragraph. Get the class to see that this is essentially a set of assumptions. Not all people in the world would be willing to accept these assumptions. Ask the pupils if they believe that all men are created equal. Raise such questions as the following: "What does this statement mean? Does the statement have the same meaning for everyone? Does Hitler believe this? Do most of us in America? Is this an assumption?" Lead the pupils to see that different people have different assumptions. Some people assume that justice and fair-play are principles which should govern human actions and others do not.

Early Proofs

In the first part of the course, most pupils do not have sufficient maturity in logical thinking to appreciate a demonstration of a relationship that appears obvious or trivial. Two suggestions are given to help overcome this difficulty:

1. **Assume more than is ordinarly done. This does not involve any sacrifice in logical rigor, provided the pupils distinguish between statements assumed and statements proved.**
2. **When demonstrating a relationship that is likely to appear obvious, emphasize the fact that we are not trying to discover whether the statement is true; we want to see whether we can show that it must follow as a necessary consequence of statements we already have.**

Congruence Theorems

It is recommended that the statements on congruent triangles be assumed, rather than proved. If this is done, the pupils should recognize that these statements are being taken for granted. They should examine them carefully, not with the idea of proving them but to make certain they are willing to accept them as statements to be taken for granted.

One method of beginning the consideration of triangles that is to lead eventually to the acceptance of statements on congruent triangles is to have the pupils draw triangles which are just like some triagles that are given. This can be done both by geometric constructions and with the use of protractors. Then raise the question, "How many parts—or what parts—of a triangle is it necessary to have given in order that the entire triangle be completely determined? Must we know all three sides and all three angles, or might something less be enough to fix the triangle completely?" Sometimes a class will reach the conclusion that any three parts, not all angles, will fix a triangle. Then the teacher must show the single case that prevents this nice, simple statement from being sufficient.

The chief objection to proving these theorems lies in the fact that they usually involve proof by superposition. The pupil has been leading up to proofs for some time, and he is likely to regard the method of proof which he uses first as something important—a pattern which he is to follow thereafter in making proofs. Then we make very little use of the method of superposition after these first theorems on congruence, and this is likely to perplex the pupil. There is nothing lost from the standpoint of logic in assuming these theorems, and it eliminates an unnecessary difficulty for the pupil.

Relating Thinking in Geometry to Thinking in Everyday Affairs

If we are to achieve the goal of developing ability to think clearly, that goal must be kept before the pupils and they must be able to see how the work of the course contributes toward the realization of that goal. Near the end of the first semester or early in the second semester, it might be well to summarize what has been learned concerning careful thinking, with emphasis upon relating thinking in everyday affairs to thinking in geometry. Suggestions are given here for studying the following points both in nongeometric and in geometric affairs:

1. The Importance of Defining Key Words and Phrases
2. Generalization
3. Reasoning by Analogy
4. Detecting Implicit Assumptions
5. Inverses, Converses, etc.
6. Indirect Proof
7. Name Calling

The teacher might direct the attention of the pupils toward the objectives of the course by means of a discussion such as the following:*

* It probably would be better to have the pupils participate in the discussion to a greater extent than is indicated in the examples given here. Some of the discussion may be elicited from the pupils by raising questions and then by carefully guiding the discussion.

"I hope that by now it is clear to all of you that the chief purpose of this course is to teach you how to think carefully. Of course we spend most of our time studying geometric relations, and it is important that you know these. A familiarity with these relationships will help you to understand and appreciate better the world we live in, and many of you will actually need to use principles of geometry in other studies or in work that you may be doing when you get out of school.

"But let's get back to the main purpose of this geometry course. It is to help you to learn how clear thinking may be done. If this were not true, we would just learn the statements of all of the theorems and study their meanings and uses and not bother at all with any proofs. For instance, one doesn't have to be able to prove the theorem on the sum of the angles of a triangle in order to know what it is and to be able to use it. In fact, some of the important geometric relationships were known and used for hundreds of years before the Greek philosophers thought of proving them. The main reason for our going through proofs is to study patterns of thought so that you may learn how clear thinking can be done. We make a list of assumptions and then see what they lead to. We find that certain other statements (we call them theorems) have to follow as a necessary result of these assumptions. When we show that a statement has to follow as a result of others, we say we are proving that statement.

"Now, as I have just pointed out, theorems don't have to be arrived at in this way. We could examine figures and I could just tell you what relationships exist. That is, I could just tell you what the theorems are and you could get acquainted with them in that way. And it would probably save time, too. We sometimes spend a day or two proving a statement which you could just take without bothering about proof and have in much less time. Then we could have more time to spend in seeing how these geometric principles are used in such practical work as surveying, navigation, building, and art work.

"Perhaps that sounds to you like it would be a good way to go through this course. The reason we don't do it that way is that the logical treatment of geometry gives us an excellent opportunity to examine and study the processes of clear thinking. If you aren't learning something about straight thinking and if you aren't becoming more careful and more critical in your thinking, then we are wasting a lot of time and we are missing the main purpose of this course.

"We are now going to consider several factors that are important in clear thinking in everyday life, and we will see if the work we have done in this course has helped us to understand these better."

1. *The Importance of Defining Key Words and Phrases*

The following procedure for leading into a discussion of the first point may be suggestive:

"The first thing we shall examine is the need of careful definition of certain words or terms. One common cause of misunderstanding is the fact that the same word may have different meanings to different people. As an example, let me raise the question, 'What is a restaurant?' "*

Some pupils quickly reply that a restaurant is a place where you can get things to eat, a place where people go to eat meals, etc. The teacher appears to be in complete agreement and continues.

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* Most of the material contained in the example on the definition of a restaurant is from *The Nature of Proof* by H. P. Fawcett. This is the *Thirteenth Yearbook* of the National Council of Teachers of Mathematics.

"Yes, all of you know what a restaurant is. You wouldn't mistake a restaurant for a hardware store or for a barber shop. It seems simple, doesn't it? But let's go into this matter a little farther. In another state a short time ago there was a great deal of controversy over just what a restaurant is. There were certain state laws that applied to restaurants and it was necessary that the word restaurant be defined carefully. Well, there was much argument over this point and different definitions were proposed. The State Restaurant Association proposed a definition of a restaurant as *a place of business where 50% or more of the gross sales are derived from the sale of foodstuffs consumed on the premises.*"

The following questions might be considered under this definition of a restaurant:

1. The gross sales of a certain hamburger lunch shop are approximately $10,000 a year, all of this coming from the sale of food. Some of this food is eaten there at the lunch shop while the remainder is taken out to be eaten elsewhere. If the amount taken out to be eaten elsewhere is $5,260.30, should this lunch stand be considered a restaurant?
2. The major part of the gross sales of the A. & P. stores is foodstuffs. Should these stores be considered restaurants?
3. Would it be possible that under the above definition some drug stores would be classed as restaurants while others, although they serve meals, would not be classed as restaurants?
4. Discuss the problem as to whether or not a place which sells beer and liquor and also serves sandwiches as well as other foods, is a restaurant.
5. Suggest some other questions or problems that might arise in this case concerning the definition of a restaurant.

Another example of a situation in which it is necessary to define key words is the following:*

Telephone rates on residence phones are lower than those on business phones. What is a business phone? Is it likely that the term business phone would have the same meaning to everyone? One definition of a business phone is *any telephone used habitually by an individual or an organized group for gain or advancement.* Under this definition, can we say for certain whether telephones in the following places are business phones: the high school; a physician's home; a physician's office; a hotel; the charity ward of a hospital; Mrs. Black, who washes for a living; Y.M.C.A.; a church; a courtesy phone at the city library?

The following are other examples:

We often hear people talk about the necessities of life. Do you think the expression "necessities of life" means the same thing to everyone? If we were going to use the expression intelligently in a discussion and we wanted to be as clear as possible, wouldn't it be better if we stated exactly what the expression was to mean in that discussion?

What words in this sign need to be understood clearly: Adults 35 cents, Children 20 cents?

A guarantee reads, "The manufacturers guarantee this appliance to be free from any defects, and they further guarantee that it will render satisfactory service for a period of one year from date of sale." If this is to be understood, what word or words need to be defined carefully?

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* This example was suggested by Miss Minnie Stewart of Topeka, Kan.

In the courtroom the judge instructs the jury to find the defendant guilty if it believes that the evidence has shown beyond reasonable doubt that he committed the crime. What words here need to be defined?

The pupils should understand that in geometry it is very necessary that certain key words be carefully defined in order that these words will have exactly the same meaning to everyone. The following are just a few examples: parallel, perpendicular, alternate interior angles, exterior angle of a triangle, isosceles triangle, congruent triangles, parallelogram, and regular polygon.

The teacher might point out to the pupils that the study of geometry ought to help them to realize the importance of carefully defining important words, and that this is an essential factor in clear thinking whether it is in geometry or in everyday affairs.

2. *Generalization*

A kind of thinking which is quite common is that of generalizing—of concluding that a thing is always true because it has been observed to be true in a limited number of cases. Generalizing is at the bottom of much loose thinking, and yet generalizing is also a very valuable method of thinking. Often the thinking which leads to important discoveries is full of generalization. The geometry pupils should become acquainted with this method of thinking and they should know something of its value and its limitations.

We cannot arrive at all of the conclusions in our daily thinking as a result of carefully thought out proofs. Many of our conclusions are generalizations. When we are guided by such conclusions we ought to recognize that we cannot be absolutely sure that they are correct, and that at best they are only *probably* true. Some examples may help to illustrate this:

John said, "I have been watching this carefully in my classes for several years and I have decided that boys are better than girls in mathematics and girls are better than boys in languages." How did John reach this conclusion? Can he be sure that his conclusion is correct? If we could consult records of large numbers of pupils could we be more confident we were reaching a correct conclusion? Could we ever be absolutely sure?

"People in this city are poor drivers. I saw two accidents yesterday, and look at that fellow there now going down the wrong side of the street." On what basis is this conclusion reached? Would you call this a good use of the method of generalizing? If we were to gather more evidence which showed that people in this town are poor drivers, would we be more certain that this conclusion is true?

In a discussion concerning family resemblances, Mary said, "I resemble my father, while my brother looks like our mother. I know of other families where that is true, too, so I think that in general girls are more likely to resemble their fathers and boys their mothers."

When a doctor attempts to determine just what is the matter with a patient, he observes all of the symptoms he can, and then he bases his diagnosis upon the experiences of other patients who have had the same symptoms.

Dogs like to ride in automobiles.

Most people do not like spinach.

Suggest other conclusions reached by generalization—some which seem to be justified and some which appear to be not very carefully thought out.

The teacher should use such examples as these to help the pupils to understand the following principles: Generalization often is a useful method of reaching conclusions. People frequently generalize too hastily and jump at conclusions that are not warranted. We can never be absolutely certain that results obtained by generalization are true. Other things being equal, the larger the number of cases a conclusion is based upon, the more certain we can be of the conclusion.

While the method of generalization is not used in geometry to prove relationships, it is used widely in establishing tentative conclusions in regard to possible or probable relationships. Anyone who is wide awake and who attempts to understand and explain things does a great deal of generalizing. The following examples may be used in a discussion of generalization in geometry.

An eighth grade boy who had just learned how to measure angles with a protractor measured the angles of a triangle and found that their sum was approximately 180 degrees. Then he measured the angles of another triangle and still another and he found in each case that the sum of the angles was approximately 180 degrees. Do you think he was doing a good piece of thinking when he concluded, "Why, the sum of the angles of any triangle must be 180 degrees!" Was he generalizing? Did he prove this conclusion for all triangles? If he were to measure the angles of several more triangles would he be justified in feeling more confident of his conclusion? Could he be absolutely sure of his conclusion?

A geometry student, upon learning that every triangle has three altitudes, sketches in the three altitudes of a triangle. It appears to him in this particular sketch that the three altitudes come very near to passing through the same point. He becomes curious and wonders if this might always be true in all triangles. (Is he generalizing?) He constructs the altitudes very varefully in two or three triangles and when he sees that the altitudes in every case pass through the same point he concludes that this relationship always holds. Can we say that the process of generalization played an important part in his discovery of this relationship? Have his experiments proved that the relationship *must* hold?

Another geometry student, Henry, concludes that since the converses of some statements are true, the converses of all statements must be true. What can you say about this conclusion? If you could show Henry a single statement whose converse is not true, would that be sufficient to destroy the validity of his conclusion?

3. *Reasoning by Analogy*

Reasoning by analogy is reasoning by comparison. It consists of comparing two things and reaching the conclusion that because they are alike in some respects they must be alike in certain other respects. It is a common form of reasoning and, like generalization, it can be used profitably at times but one must be cautious about accepting the conclusions. The following examples may help to illustrate this kind of reasoning:

In an advertisement a woman is quoted as saying that Smith Brothers Cough Syrup gave her relief from a cough and that within three hours she had lost the cough completely. The advertiser wants us to think that the medicine will do the same thing for us if we have a cough. Do you think there is sufficient reason given to justify this conclusion?

Another advertisement quotes a famous movie actress as saying that she uses a certain kind of soap and that it is good for her complexion. Discuss.

In the field of medicine, many newly discovered drugs are tried out on animals such as guinea pigs before being used with people. Guinea pigs seem to react like human beings to many medicines. If a drug is to be tested as a cure for a certain disease, the disease is given to some guinea pigs and then the cure is tried out on them. Discuss the reasoning behind this practice.

Mary said one morning, "I believe it will be colder this afternoon than it was yesterday afternoon. It is bright and clear this morning like it was yesterday morning, and it is colder this morning than it was yesterday morning. The wind is the same as it was yesterday morning, too. Has Mary thought this matter over pretty carefully? Is she reasoning by similarity (by analogy)? Can she be absolutely certain that her conclusion is correct?

In geometry, reasoning by analogy furnishes many suggestions concerning relationships that may exist. However, we should be careful about accepting the conclusions of this kind of thinking. We can never definitely prove anything by this method. Some examples of reasoning by analogy follow:

In proving that the angles opposite equal sides in a triangle are equal James worked out a proof in which he made use of a construction line to divide the given triangle into two triangles which he could prove congruent. Later, when he wanted to prove that the sides opposite equal angles in a triangle are equal, the similarity of the two situations led him to suspect that he might make use of a similar constructions line in this proof. He did so and was able to prove the theorem. Was this a good use of thinking by similarity?

In the early part of the geometry course, after a class has found that two triangles are congruent if the three sides of one are equal to the three sides of the other, some of the pupils think that they should be able to prove that two triangles must be congruent if they have the three angles of one equal to the three angles of the other.

When the geometry class began a study of quadrilaterals, John reasoned that since two triangles with corresponding sides equal are congruent, two quadrilaterals ought to be congruent if they have all four sides of one equal to the corresponding sides of the other.

4. *Detecting Implicit Assumptions*

If one is to become truly critical in his thinking he not only must be able to follow the steps in an argument, but he must be able to see the assumptions upon which the argument is based. Usually these assumptions are not stated and it requires pretty clear thinking to get back of a discussion or an argument and see what these assumptions are. Many of our actions and decisions would be more consistent if we would stop occasionally and check up on the assumptions which influence much of our behavior. We probably very seldom analyze our beliefs.

Exercises similar to those beginning at the bottom of page 6 and continuing to the top of page 8 in this pamphlet may be used to give the pupils practice in detecting hidden assumptions.

This is a good place to examine the processes of generalization and reasoning by analogy to find the assumptions upon which they are based. Generalization involves the assumption that the unobserved cases will be just like those from which the generalization is made. Reasoning by an-

alogy is based upon a similar assumption: if two cases are alike in some respects they must be alike in other respects. In a sense, reasoning by analogy is a form of generalization.

In geometry we frequently find pupils making use of assumptions without recognizing that they are using them. It would be well to call the attention of the class to this danger and to point out that this also is a common fault in everyday thinking. The following examples illustrate this error in geometry:

> Pupils sometimes assume the thing they are attempting to prove. Thus, a pupil who is attempting to prove that two lines are parallel may say that a transversal cutting these lines forms a pair of equal alternate interior angles.
>
> A pupil states the converse or the inverse of a previous theorem and thinks that this statement can be used logically in support of a step in his proof. (This point is discussed further in the discussion of converses and inverses.)
>
> A common error is that of drawing conclusions based upon the appearance of the figure. This involves the use of assumptions which are not recognized and stated. For example, an angle may appear to be a right angle or a triangle may appear to be isosceles, and the pupil uses these as facts although they are not contained in the statement of the theorem.

5. *Inverses, Converses, Etc.*

The whole subject of inverses, converses, and contrapositives is an interesting one, and it deserves a more complete treatment than it is given here. Teachers may become better acquainted with the possibilities of this field by referring to Dr. Nathan Lazar's thesis, *Logic in Geometry*, or to his articles in the March, April, and May, 1938, issues of *The Mathematics Teacher*.

It should be said at the outset that the teacher ought to be aware of the inaccuracy of speaking in general of *the* converse of a statement, for many statements have more than one converse. (Lazar proposes the definition: A converse of a proposition may be obtained by interchanging any number of conclusions with an equal number of hypotheses.) The teacher also should have clearly in mind what is meant by an inverse (or opposite) of a statement. Here, again, it should be recognized that it is incorrect to refer in general to *the* inverse (or opposite) of a statement, because many statements have more than one inverse. (See Lazar for a careful definition of inverses.)

We may say here that for a statement which has a single hypothesis and a single conclusion:

> The converse may be obtained by interchanging the hypothesis and the conclusion.
> The inverse may be obtained by denying both the hypothesis and the conclusion.

Thus the converse of the statement, "all Buicks are good cars," is "if a car is a good one it is a Buick," and the inverse is "a car that is not a Buick is not a good car."

Both converses and inverses are common in everyday thinking, and too often they are not recognized. We sometimes accept a particular state-

ment; and then, without really being aware of it, we infer a converse or an inverse of that statement.

It probably would help the pupils to recognize converses and inverses if they would write several statements (rather simple ones) and then give for each one: its converse, its inverse, the converse of its inverse, and the inverse of its converse. The following list may be suggestive:

If a driver is speeding, he is breaking the law.
The A and P stores sell standard goods at low prices.
The definitions in Macmillan's Modern Dictionary are up-to-date.
Senator Blank's plan is a plan which would aid the farmers.
If two angles of a triangle lie opposite equal sides, they are equal.
One must have influential friends to get a political job.
If the corresponding sides of two triangles are equal, the corresponding angles must also be equal.

6. *Indirect Proof*

Of the various forms of indirect proof, only proof by exclusion is treated here. This method of proof is exceedingly common in everyday thinking, and we should see that our pupils are able to recognize it. It is a simple form of proof. It involves, first of all, agreeing upon the number of possible conditions, and then the elimination of all but one of these.

The following example illustrates the use of indirect proof:*

While her mother was out of town for a few days, Roberta broke a vacuum bottle. Only the top remained unbroken. She was very anxious to replace the bottle before her mother's return; so she hurried downtown, taking the top of the bottle with her. The clerk was very helpful. He said, "That type of bottle comes in only three sizes. Do you know what size yours is?" Roberta did not know; so he continued, "Well, I'll tell you. We happen to have only two of the sizes in stock, but we can try those and see if one of them fits." The top did not fit either of the sizes they had in the store, and the clerk said he would order a bottle of the third size.

Many teachers dislike to use an indirect proof in the geometry class because they feel that it is a difficult and unnatural method of approaching a problem. It *is* difficult if we begin a proof in a manner similar to the following:

"We are going to prove these lines are parallel by assuming that they meet and then showing that they can't meet and therefore must be parallel."

As soon as we say that we are going to prove the lines parallel by assuming that they meet, we lose most of our class. And no wonder! The pupils do not see the method of proof as a pattern of thought. It all sounds mysterious or even ridiculous. But suppose we explain our procedure something like this:

"Would we all agree that these two lines either meet or they are parallel? that there is no other possibility? that they must be one or the other? Our definition of parallel lines would justify that statement, wouldn't it? All right, then, if we can

* This example was suggested by one of the workers at the Denver Workshop of the Progressive Education Association, in the summer of 1938.

rule out one of these two possibilities, the other one would be the only one left. We are going to see if we can rule out the possibility that the lines meet. Suppose they *did* meet—let's see what would happen."

The pupil should see the pattern of thought, and he should come to know it as one which is used frequently in non-geometric situations. He should realize that one of the essential points to watch in its use is that all of the possibilities are considered.

Christofferson* has suggested that indirect reasoning is commonly used (1) in finding a book in a library, (2) in locating a boy in a large high school, (3) in finding an address in a classified directory, (4) in deciding on the meaning of a word in a certain setting, (5) in repairing a car, (6) in planning a meal, (7) in finding John in the evening, (8) in deciding what bait to use in fishing, (9) or what club to use in golfing, (10) or what sort of return to give your opponent in tennis, (11) or what sort of a ball to pitch to a certain batter in baseball, or (12) what play to use in football, or (13) what dress to wear to a party.

7. *Name Calling*

Although name calling is not ordinarily encountered in geometry, it is such a common device of propaganda that it should be called to the attention of pupils who are studying clear thinking. Name calling is used in an attempt to influence our judgment without presenting the basic evidence. A political speaker often will devote most of a speech to calling his opponent names and attempting to ridicule him by means of cleverly worded references to some of his acts. These references will be filled with catch phrases designed to make the audience scorn or laugh at the opponent. One clue which may be used to detect this device is the lack of a carefully stated argument.

Methods of Leading the Pupil to Discover Relationships for Himself

If we examine the kind of thinking that is being done on all sides of us, we cannot escape the conclusion that much of it is rationalization. People spend far more time and energy in trying to justify a position they have taken than they do in thinking their way through to logical conclusions. We are much more concerned with confirming our prejudice and trying to convince other people that our stand is the right one than we are with striving to reach a position which is logical and consistent.

Yet this is precisely the kind of thinking we have been emphasizing in our geometry classes. The pupil is told what position to take. He is handed a statement and asked to defend it. He has little opportunity to see where it comes from.

We go through the course asking him to prove this and to prove that.

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* Christofferson, H. C., "Geometry a Way of Thinking," *Mathematics Teacher* XXXI: 147-55; April, 1938.

We often begin a lesson by saying, "Today we are going to prove the theorem. . . ." The pupil is not expected to be concerned with the discovery of these relationships or the wording of statements describing them. All of his energies are directed toward defending—proving—these statements.

It is small wonder that so few pupils are able to see that in geometry we are simply examining the consequences of statements which we have agreed to take for granted. The pupils do not have an opportunity to search for and to discover these consequences. We have them all carefully worded and give them out to the pupils, who are then required to build proofs for them.

Would it not be better to have the pupils arrive at these relationships through a search for the implications of their set of assumptions? The teacher could direct this search by raising questions concerning the possibility of certain relationships being implied by those that have gone before. Sometimes this could lead to exercises for the pupils to solve, and at other times it could lead to theorems. Incidentally, some relationships should be suggested which may appear at first to be valid but which actually are not valid or are special cases.

In following this procedure, the pupil first would be led to suspect a relationship, then he would investigate its validity, and finally he would demonstrate it. Notice that the proof is the *final* step. The thinking involved in reaching a conclusion often follows a pattern which is quite different from that which is finally used in demonstrating the conclusion. If an individual is not permitted to have some part in seeking for and testing conclusions, he probably will not be able to appreciate the refinement of demonstrations. Moreover, he is being deprived of the thrill of discovering things for himself. If this approach is to be followed, it is essential that the pupil have the attitude that he is seeking relationships that must follow from others, rather than that he is proving relationships which are given to him.

It probably would be impossible or at least impracticable to lead the pupil to discover all of the theorems and other relationships for himself. However, if he discovers even a few, he probably will be in a better position to see the nature of the whole structure and method of the science of geometry.

The following methods are suggested for leading the pupil to discover theorems for himself:

1. Experimentation with ruler and compasses.
2. Examination of converses already proved or assumed.
3. The principle of duality.
4. A study of locus problems.
5. A consideration of what takes place when a figure undergoes a continuous change.

The first method may be employed effectively throughout most of the course. The pupils can be led to experiment and to hunt for possible re-

lationships in attempting to answer such questions as the following, which may be raised by the teacher:

If the diagonals of a quadrilateral bisect each other, must the figure be a parallelogram?

In what kind of a quadrilateral are the diagonals always equal?—or perpendicular to each other?

If a line bisects an angle of a triangle, must it bisect the side opposite?

If a line passes through the mid point of one side of a triangle and is parallel to one of the other two sides, do you think there may be some other relationship which will always be present?

In leading up to the theorems on concurrent lines of a triangle the teacher might have the class sketch or construct the altitudes of some triangles. This might be done for the apparent purpose of calling attention to the fact that a triangle has three altitudes. Probably someone will ask whether the altitudes always go through the same point. If not, the teacher can draw the question by asking, rather incidentally, if the pupils suspect anything about the altitudes. Then the teacher might ask the class to suggest other sets of lines in a triangle. They may suggest other lines than those we want, but they will surely suggest the perpendicular bisectors of the sides and the bisectors of the angles.

If the pupil once attains the viewpoint that he is searching for relationships which he can investigate, it is not difficult to get him into the habit of examining converses of statements. As soon as a relationship is established, it converse (or its converses) should be examined. Such an examination will not always prove fruitful, but it will often yield interesting relationships. It will also contribute toward the pupil's understanding of the nature of converses as well as being an aid in getting him to feel that he is playing some part in finding the statements which he proves.

The principle of duality furnishes another source of possible relationships to be investigated. The pupil's attention can be called to the dual relationship between point and line. This can be seen in the assumptions: "Two points determine a straight line," and "Two lines which are parallel intersect in (or determine) a point." It can also be seen in two of the congruence theorems, either of which may be obtained from the other by interchanging the words "angle" and "side." Whenever a relationship is established which deals with the sides of a triangle, we can look to see if there could be a corresponding relationship dealing with the angles. An example is the concurrence of the perpendicular bisectors of the sides of a triangle and the concurrence of the bisectors of the angles.

The fourth and fifth suggestions in the list might have been included under the first, but we thought they merited listing separately. In some cases a clue to a relationship is obtained by considering the location of all points which satisfy a given condition. In other cases it frequently helps, in examining a suspected relationship, to allow the figure to change its size and shape, and to see whether the relationship remains throughout such changes.

A Final Word

From the standpoint of postulational thinking it is unfortunate that reference has so often been made to the great truths of geometry. It is difficult to get individuals to understand the if-then type of thinking if conclusions are regarded as belonging to the absolute, unchanging truths of the universe. For this reason it is strongly recommended that teachers avoid referring to theorems as the truths of geometry. In place of a question like, "Do you think such and such a relationship is true?" teachers might substitute, "Do you think we can show that it must follow?" The pupils should regard the theorems of geometry merely as the necessary conclusions of a set of assumptions. They should understand that even a slight change in the set of assumptions may produce changes in the conclusions drawn from these assumptions. Before the year is ended the pupils should know that there are geometries in existence today other than the one they are studying. They should know something of the origin of these geometries, that they differ from Euclidean geometry only because of slight differences in a single assumption, and that the mathematicians and physicists today employ the Riemannian geometry because they believe that it describes our universe better than does Euclidean geometry.

It should be pointed out that teachers who have used some non-geometric materials in the manner suggested in this pamphlet report that there is no loss in the pupils' understanding of geometry because of the time spent on non-geometric material. They report, on the contrary, that pupils gain in their understanding of geometric relationships, because the pupils use in geometry those principles of clear thinking which they have learned. Their thinking is more effective, in geometry as well as elsewhere.